# Political power on a line graph

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**Abstract**

We consider situations of majority voting, where the players are ordered linearly. This order may be based on, for example, ideology or political preferences over economic policy, ethical principles, environmental issues, and so on. Winning and losing coalitions are given by a majority voting game, while restrictions on cooperation are determined by a line graph, where only connected coalitions are feasible and form a (winning) coalition. Various solutions for line-gaph games can then be viewed as power indices measuring the ability of political parties to turn losing coalitions into winning ones, taking into account the cooperation restrictions among the parties. Here, we start by observing that a number of existing power indices either are not core stable, or do not reward intermediate veto players. Then, we take a closer look at the average hierarchical outcome, called hierarchical index in the context of this paper, and the $\tau$-index. These indices are core stable and, moreover, reward all veto players. Specifically, the $\tau$-index rewards all veto players equally, while the hierarchical index always assigns higher power to the two extreme veto players than to intermediate veto players. We axiomatically characterize the (i) hierarchical index by core stability and a weaker version of component fairness, and (ii) the $\tau$-index by core stability and a weaker version of Myerson's (1977) fairness property.

Keywords: majority voting game, line-graph game, power index, core stability, hierarchical index, $\tau$-index

# 1 Introduction

In this paper, we will be concerned with measuring voting power in specific types of *majority voting situations*. Among the many existing power indices in the literature, some of the most famous ones are the *Shapley-Shubik index* (Shapley and Shubik, 1954), which is equivalent to the *Shapley value* (Shapley, 1953) applied to the associated voting game, and the *Banzhaf-Penrose index* (Banzhaf, 1965; Penrose, 1946). By the voting power of a given political party, we mean its ability to effect a change in the outcome: turning losing coalitions into winning ones, or vice versa. To determine these abilities, it is sufficient to know the winning and losing coalitions in a voting situation. These coalitions can be summarized by an associated *majority voting game*: a cooperative or coalitional game where the worth of any coalition is either one (when it has the required qualified majority and is thus winning) or zero (when it does not have the required qualified majority and is thus losing).

The most famous set-valued solution for cooperative games is the *core* (Gillies, 1953), which, for any game, is the set of all efficient payoff vectors such that every coalition earns at least its own worth. In the context of voting games, we will mostly refer to payoff vectors as *power* vectors. For a majority voting game, this means that, in every core payoff vector, the payoffs or powers of the parties in a winning coalition must sum up to one. Specifically, the powers of *all* parties must sum up to one since the grand coalition—the set of all players in the game—is winning. Then, it immediately follows that the core of a voting game is non-empty if and only if the game has veto players: players who belong to every winning coalition. In that case, the core consists of all allocations where the full power of one is allocated over the veto players, and where non-veto players are assigned a power of zero.

In a cooperative game, any coalition of players can form and obtain its worth. In voting games, this means that any coalition of parties can cooperate and try to form a majority or winning coalition. However, in real-life politics, not every combination of parties can form a coalition. In some cases, parties exclude each other from forming coalitions. What's more, even when parties do not exclude each other, it might be that two parties can only belong to the same coalition if another (ideologically intermediate) party belongs to that coalition as well. One way to model such cooperation restrictions is by using Myerson's (1977) (communication) graph game model. In this model, the players in a cooperative game are also the nodes in an undirected graph such that two players are linked if and only if they can cooperate together without any other player. The feasible coalitions in that case are the connected coalitions in the graph. Various existing solutions take account of these cooperation restrictions in allocating the payoffs over the players. One of the first such solutions was introduced by Myerson (1977) and is obtained by applying the Shapley value to the so-called restricted game. This restricted game is obtained by assigning to every coalition the sum of the worths of its maximally connected subsets (components). This solution was later called the *Myerson value*. Myerson (1977) characterized this solution as the unique solution for communication graph games that satisfies *component efficiency* and *fairness*. Component efficiency requires that the sum of the payoffs of all players in a maximally connected subset (that is, component) is equal to the worth of that component. Fairness requires that deleting an edge between two players has the same effect on their payoffs.

For cycle-free graph games, Demange (2004) introduced the concept of *hierarchical outcomes*, while Herings, van der Laan, and Talman (2008) considered the average of all hierarchical outcomes and characterized this solution by component efficiency and *component fairness*. Component fairness requires that deleting an edge between two players in a cycle-free graph game has the same effect on the per capita payoffs in the two newly created components (each containing one of the two players whose link is broken). Béal, Rémila, and Solal (2010) considered weighted combinations of hierarchical outcomes. Hierarchical outcomes are special marginal vectors of the game where the order in which the players enter the grand coalition is restricted by the graph. An interesting property of the hierarchical outcomes (and their convex combinations) is that they always belong to the core of the restricted game if the game is superadditive and the graph is cycle-free. Since majority voting games are superadditive and line graphs are cycle-free, this immediately implies that the hierarchical outcomes and their convex combinations assign core power vectors when we restrict a majority voting game to a line graph. In van den Brink, van der Laan, and Vasil'ev (2007), the special class of line-graph games is considered and particular attention is paid to two hierarchical outcomes (the so-called upper equivalent solution and the lower equivalent solution) and their average. These outcomes also regularly appear in the applied economics and operations research literature. Specifically, the *upper equivalent solution*, which assigns to every line-graph game the *marginal vector* where the players enter from left to right, yields the downstream incremental solution for river games in Ambec and Sprumont (2002) or the drop out monotonic solution for sequencing games in Fernández et al. (2005). The *lower equivalent solution*, which assigns to every line-graph game the marginal vector where the players enter from right to left, coincides with the upstream incremental solution for river games in Ambec and Ehlers (2008). The average of the upper and lower equivalent solutions yields the equal gain split rule introduced for one-machine sequencing games by Curiel et al. (1993, 1994).

In this paper, we consider majority voting games where the political parties are ordered on a line according to their political preferences over, for example, economic policy, ethical issues, environmental problems, and so on. We apply various solutions to the associated line-graph games as a way of measuring the parties' power, taking into account their positions on the line. The upper equivalent solution mentioned above assigns full power to the pivotal party in a *majority voting line-graph game* when the parties enter from left to right. The lower equivalent solution assigns full power to the pivotal party in a majority voting line-graph game when the parties enter from right to left. These two parties are also the most right-wing, respectively left-wing, veto players. Applying the average of the upper and lower equivalent solutions assigns equal (half) power to the left and right-wing pivotal party.

Since all hierarchical outcomes and their convex combinations belong to the core of a majority voting line-graph game, it must hold that these two hierarchical outcomes, and their average, fully allocate power over veto players and give zero power to non-veto players. This is an important difference with the Shapley value (Shapley-Shubik index (1954) for voting games) or the Banzhaf value (Banzhaf, 1965) which, when applied to the restricted game of a majority voting line-graph game, ascribe positive power to non-veto players, and thus do not belong to the core of the restricted game.

The goal of this paper is to find core solutions that also reward non-extreme veto players. Whereas the average of the upper and lower equivalent solutions (the equal gains split rule) allocates the full power of one equally over the two extreme (left and right-wing) veto players, other combinations of hierarchical outcomes allocate the full power over *all* veto players, assigning 'intermediate' or non-extreme veto players some positive power. Typically, however, in the average of all hierarchical outcomes, the two extreme veto players get the highest power. We refer to the power index that assigns to every majority voting line-graph game the average hierarchical outcome as the *hierarchical index*. We then axiomatically characterise this power index by core stability and a weaker version of Herings, van der Laan, and Talman's (2008) component fairness property, where we only consider deleting edges between veto players.

An interesting power index in this context is the one that allocates the full power *equally* over all veto players. In this paper, we will show that this rule is obtained by applying the *$\tau$-value* (Tijs, 1981) to the restricted game of a majority line-graph voting game. We call the resulting power index the *$\tau$-index*. Moreover, we axiomatically characterize this power index by core stability and a weaker version of Myerson's (1977) fairness property, where again we only consider deleting edges between veto players.

This paper is organized as follows. In Section 2, we discuss preliminaries on majority voting games, line-graph games, and solutions on line-graph games. In Section 3, we apply known results on line-graph games to the measurement of political power on majority voting line-graph games. In Section 4, we consider the hierarchical outcomes and the $\tau$-value as measures of political power on line-graph games and provide an axiomatization. Section 5 contains concluding remarks.

# 2 Preliminaries

## 2.1 Cooperative games

A situation in which a finite set of players can obtain certain payoffs through cooperation can be described by a *cooperative game with transferable utility* or, simply, a cooperative game. A cooperative game is a pair $(N, v)$, where $N = \{1, \ldots, n\}$ is a finite set of $n$ players, and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on $N$ such that $v(\varnothing) = 0$. For any coalition $S \subseteq N$, the real number $v(S)$ is the *worth* of coalition $S$; that is, the members of coalition $S$ can obtain a total payoff of $v(S)$ by agreeing to cooperate.

In this paper, we assume that $N$ is fixed. This allows us to refer to a cooperative game $(N, v)$ simply by its characteristic function $v$. We denote the collection of all cooperative games on $N$ (represented by their characteristic function) by $\mathcal{G}^N$. We first recall some properties of cooperative games. A cooperative game $v$ is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for any pair of subsets $S, T \subseteq N$ such that $S \cap T = \varnothing$. Further, a cooperative game $v$ is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. Every convex game is superadditive. A special class of convex games are unanimity games. For each non-empty $T \subseteq N$, the *unanimity* game $u_T$ is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. It is well-known that unanimity games form a basis for $\mathcal{G}^N$. Specifically, every game $v$ can be expressed as a unique linear combination of unanimity games,

$$v = \sum_{S \subseteq N,\, S \neq \varnothing} \Delta_S(v) u_S,$$

where $\Delta_S(v)$ are *the Harsanyi dividends* (see Harsanyi, 1959), given by

$$\Delta_S(v) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T), \quad S \subseteq N, S \neq \varnothing. \tag{2.1}$$

Equivalently, by applying the Möbius transformation, we have

$$v(S) = \sum_{T \subseteq S} \Delta_T(v), \quad S \subseteq N, S \neq \varnothing. \tag{2.2}$$

So, the worth of coalition $S$ is equal to the sum of the dividends of all subcoalitions of $S$. This also gives a recursive definition of the Harsanyi dividends. The dividend of every one-player coalition is equal to its worth, while, recursively, the dividend of every coalition with at least two players is equal to its worth minus the sum of the dividends of all of its proper subcoalitions. In this sense, the dividend of a coalition $S$ can be interpreted as the extra benefit from cooperation among the players in $S$ that they cannot realize through cooperation in smaller coalitions.

**Solution concepts.** A *payoff vector* of a cooperative game $(N, v)$ is an $n$-dimensional vector that assigns a payoff to any player $i \in N$. A *point-valued solution* is a function $f$ that assigns a single payoff vector $f(v) \in \mathbb{R}^N$ to any game $(N, v)$. A point-valued solution $f$ is *efficient* if, for any game $(N, v)$, it distributes precisely the worth of the grand coalition: $\sum_{i \in N} f_i(v) = v(N)$ for all $v \in \mathcal{G}^N$. An example of an efficient point-valued solution is the famous Shapley value (Shapley, 1953): the average of the so-called marginal contribution vectors.[1]

For a permutation $\pi \colon N \to N$ that assigns rank number $\pi(i) \in N$ to any player $i \in N$, we define $\pi^i = \{j \in N \mid \pi(j) \leq \pi(i)\}$, that is, $\pi^i$ is the set of all players with rank number at most equal to the rank number of $i$, including $i$ itself. Then the *marginal contribution vector* $m^\pi(v) \in \mathbb{R}^N$ of game $v$ and permutation $\pi$ is given by

$$m_i^\pi(v) = v(\pi^i) - v(\pi^i \setminus \{i\}), \quad \text{for all } i \in N.$$

The vector $m_i^\pi(v)$ thus assigns to player $i$ its marginal contribution to the worth of the coalition consisting of all of its predecessors in $\pi$. The *Shapley value*, $Sh$, assigns to every game the average of the marginal contribution vectors over all permutations, and is thus defined by

$$Sh_i(v) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} m_i^\pi(v), \quad \text{for all } i \in N,$$

where $\Pi(N)$ is the collection of all permutations on $N$.[2]

Writing

$$m_i^S(v) = v(S \cup \{i\}) - v(S),$$

as the marginal contribution of player $i \in N$ to coalition $S \subseteq N \setminus \{i\}$, the *Banzhaf value* (see Owen, 1975 and Dubey and Shapley, 1979 as an extension of Banzhaf, 1965), $Ba$, is defined by

$$Ba_i(v) = \frac{1}{2^{|N|-1}} \sum_{S \subseteq N \setminus \{i\}} m_i^S(v), \quad \text{for all } i \in N.$$

[1] For a recent survey on the Shapley value, see Algaba, Fragnelli, and Sánchez-Soriano (2019).

[2] Alternatively, for any game, the Shapley value distributes equally the Harsanyi dividend of coalition $S$ over the players in $S$.

Thus, $Ba_i(v)$ is the average marginal contribution of player $i$ to every coalition that does not contain $i$ assuming that every coalition has equal probability of occurring. The Banzhaf value is not efficient.[3]

The $\tau$-value is defined in Tijs (1981) as an efficient solution for the class of quasi-balanced games. In order to define the $\tau$-value and the class of quasi-balanced games, we first need to define two types of payoff bounds. Specifically, let the upper payoff bound of game $v$ be given by the so-called *utopia payoff vector* $M(v)$ defined as

$$M_i(v) = v(N) - v(N \setminus \{i\}), \quad \text{for all } i \in N.$$

The vector $M(v)$ thus assigns to every player their marginal contribution to the grand coalition $N$. Further, let the lower payoff bound of game $v$ be given by the so-called *minimal right vector* defined as

$$m_i(v) = \max_{S \subseteq N, i \in S} \left( v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \right), \quad \text{for all } i \in N.$$

The class of quasi-balanced games on $N$, denoted by $QB^N$, is the class of games for which $M(v)$ and $m(v)$ constitute genuine upper and lower bounds in the following sense: (1) each player's utopia payoff is at least as large as that player's minimal right, (2) all utopia payoffs sum up at least to the worth $v(N)$ of the grand coalition, and (3) all minimal rights sum up at most to the worth $v(N)$ of the grand coalition:

$$QB^N = \left\{ v \in \mathcal{G}^N \;\middle|\; m(v) \leq M(v) \text{ and } \sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v) \right\}.$$

The $\tau$-value is defined on the class of quasi-balanced games and, for every $v \in QB^N$, is given by

$$\tau(v) = m(v) + \alpha(M(v) - m(v)),$$

where $\alpha \in \mathbb{R}$ is such that the $\tau$-value is efficient: $\sum_{i \in N} \tau_i(v) = v(N)$. The $\tau$-value assigns to each player, in an efficient way, their minimal right plus a (uniform) share of the margin by which their utopia payoff exceeds their minimal right.

A *set-valued solution* for cooperative games is a mapping $F$ that assigns to every game $(N, v)$ a set of payoff vectors $F(v) \subset \mathbb{R}^N$. The most famous set-valued solution, the

[3] Efficient normalizations of the Banzhaf value are the multiplicative and additive normalizations. The multiplicative normalization allocates $v(N)$ proportionally to the Banzhaf values of the players (see van den Brink and van der Laan, 1998). The additive normalization is the least square value (see Ruiz, Valenciano, and Zarzuelo, 1998) obtained by adding or subtracting the same amount from the Banzhaf value payoffs of the players so that an efficient payoff vector results.

*core*, introduced by Gillies (1953), is the set of all efficient payoff vectors that cannot be improved upon by any coalition; that is, any payoff vector in the core is efficient and each coalition gets at least its own worth:

$$core(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^{n} x_i = v(N), \text{ and } \sum_{i \in S} x_i \geq v(S), \text{ for all } S \subseteq N \right\}.$$

The core of a game can be empty. It is well-known that $core(v)$ is non-empty if and only if $v$ is balanced (which was shown independently by Bondareva, 1963 and Shapley, 1967).

## 2.2 Line-graph games

Line-graph games are a special class of games with communication (graph) structure studied in Myerson (1977). We may assume, without loss of generality, that a line graph reflects the natural ordering from 1 to $n$. The structure on the set of players then is given by a line graph $(N, L)$, where $N$ is the set of players and $L \subseteq \overline{L} = \{\{i, i+1\} \mid i = 1, \ldots, n-1\}$ is the set of (undirected) *edges*. Notice that $\overline{L}$ is a linear order on $N$. However, we allow for any subset $L$ of $\overline{L}$ to be a line graph; hence, a line graph can consist of disconnected parts. Let $\mathcal{L}^N$ be the power set of $\overline{L}$, that is, the set of all line graphs on $N$ given the natural ordering from 1 to $n$. We further denote the collection of all line-graph games on $N$ by $\mathcal{G}^N \times \mathcal{L}^N$. For short, we denote the game $(N, v)$ with line graph $(N, L)$ as the *line-graph game* $(v, L)$.

Following Myerson (1977) and Greenberg and Weber (1986), in a line-graph game $(v, L) \in \mathcal{G}^N \times \mathcal{L}^N$, players can only cooperate when they are able to communicate with each other. This means that a coalition $S \subseteq N$ can only realize its worth $v(S)$ when $S$ is *connected* in the line graph $(N, L)$. Clearly, for the 'full' line graph $(N, \overline{L})$, the set $\mathcal{I}$ of (non-empty) connected coalitions is given by[4]

$$\mathcal{I} = \{S \subseteq N \mid S = [i, j],\ 1 \leq i \leq j \leq n\},$$

where $[i, j]$ denotes the set of consecutive players $\{i, i+1, \ldots, j-1, j\} \subseteq N$. For any line graph $(N, L)$, the set of connected coalitions is a subset of $\mathcal{I}$ and consists of those coalitions $[i, j]$ where $i$ and $j$ belong to the same component. Connected coalition $T = [l, m] \in \mathcal{I}$ is a *component* in $L \subseteq \overline{L}$ if $\{i, i+1\} \in L$ for all $i \in [l, m-1]$ and $\{\{l-1, l\}, \{m, m+1\}\} \cap L = \varnothing$.

[4] In the more general model of Myerson (1977), the players belong to a communication structure that is represented by a *graph* $(N, A)$, where the player set $N$ is the set of nodes and where $A \subseteq \{\{i, j\} \mid i, j \in N,\ i \neq j\}$, a collection of unordered pairs, is the set of edges reflecting the communication possibilities among the players. A coalition $S \subseteq N$ can realize its worth $v(S)$ when $S$ is *connected* in graph $(N, A)$, that is, when for any two players $i$ and $j$ in $S$, there is a subset $\{\{i_k, i_{k+1}\} \mid k = 1, \ldots, t\} \subseteq A$ of edges such that $i_1 = i$, $i_{t+1} = j$, and $\{i_2, \ldots, i_t\} \subseteq S$.

For any coalition $S \subseteq N$, we denote the collection of components in line graph $L(S) = \{\{i,j\} \in L \mid i,j \in S\}$ by $C_L(S)$. When there is no ambiguity about the line graph $L$, we simply refer to this as the collection of components of $S$. Observe that the collection $C_L(S)$ of components of $S$ forms a partition of $S$.

The set of connected coalitions in line graph $L \subseteq \overline{L}$ is denoted by

$$\mathcal{I}(L) = \{[i,j] \in \mathcal{I} \mid \text{there exists a } T \in C_L(N) \text{ such that } i,j \in T\}.$$

In the restricted game introduced by Myerson (1977) (for arbitrary graph games), a connected coalition earns its worth, but when coalition $S$ is not connected, the players in $S$ can only realize the sum of the worths of its components. So, for a given line-graph game $(v,L)$, the restricted game $v^L \in \mathcal{G}^N$ induced by line graph $(N,L)$ is given by[5]

$$v^L(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{I}(L), \\ \sum_{T \in C_L(S)} v(T), & \text{if } S \notin \mathcal{I}(L). \end{cases} \tag{2.3}$$

We defined a line graph $L$ to be any subset of the (complete) linear order $\overline{L}$. But, notice that restricting the restricted game $v^L$ further on the linear order $\overline{L}$ does not have an impact: $v^L = (v^L)^{\overline{L}}$.

Applying a formula stated in Owen (1986) for cycle-free graph games,[6] gives the following useful expression for the Harsanyi dividends in line-graph games.

**Theorem 2.1** *[Owen (1986), Bilbao (1998), van den Brink, van der Laan, and Vasil'ev (2007)[7]] Consider line-graph game $(v,L)$. Then, the dividends of the restricted game $v^L$ are given by*

$$\Delta_S(v^L) = \begin{cases} 0, & \text{if } S \notin \mathcal{I}(L), \\ v(\{i\}), & \text{if } S = \{i\}, \\ v[i,j] - v[i+1,j] - v[i,j-1] + v[i+1,j-1], & \text{if } S = [i,j] \in \mathcal{I}(L), j > i. \end{cases} \tag{2.4}$$

This implies that the dividend of any coalition $S$ is fully determined by the worths of at most four coalitions, irrespective of the size of $S$. In contrast, for general cooperative games, the Harsanyi dividend of coalition $S$ depends on the worths of all $2^{|S|}$ subsets of

[5]For definitions on arbitrary graph games, we refer to Myerson (1977).

[6]See also Bilbao (1998) for the more general class of cycle-complete graphs.

[7]In van den Brink, van der Laan, and Vasil'ev (2007), the last line of (2.4) is shown to hold for $\overline{L}$. Applying it to $(v^L)^{\overline{L}}$ yields, for $[i,j] \in \mathcal{I}(L)$ and $j > i$, $v^L[i,j] - v^L[i+1,j] - v^L[i,j-1] + v^L[i+1,j-1]$. This expression reduces to (2.4) since $v^L(S) = v(S)$ and $\Delta_S(v^L) = \Delta_S(v^{\overline{L}})$ for any connected coalition $S \in \mathcal{I}(L)$.

the coalition. Expression (2.4) turns out to be very useful for applications, as we will see in Section 3 where we discuss majority voting line-graph games.

It is well-known that the restricted game of a superadditive game on the complete line graph $\overline{L}$ is balanced (see, for example, Le Breton, Owen, and Weber, 1992; Demange, 1994; and Potters and Reijnierse, 1995). This result follows immediately from Granot and Huberman (1982), who showed that a so-called permutationally convex game is balanced. More precisely, let $u$ and $\ell$ be the two permutations on $N$ defined by

$$u(i) = i, \; i = 1, \ldots, n,$$

respectively,

$$\ell(i) = n + 1 - i, \; i = 1, \ldots, n.$$

When $v$ is superadditive, the restricted game $v^{\overline{L}}$ satisfies the permutational convexity condition of Granot and Huberman (1982) for the two permutations $u$ and $\ell$. Further, it then follows that the two marginal vectors $m^u(v^{\overline{L}})$ and $m^\ell(v^{\overline{L}})$ are in the core of $v^{\overline{L}}$ (see also Demange, 2004). Since $v^L$ is superadditive for any superadditive game $v$ and any line graph $L \subseteq \overline{L}$,[8] these results also hold for any superadditive game $v$ restricted to a line graph $L \subseteq \overline{L}$.

**Properties.** We recall some properties of solutions for line-graph games. First, component efficiency requires that the sum of the payoffs of the players in any component equals the worth of that component.

- A point-valued solution $f$ on $\mathcal{G}^N \times \mathcal{L}^N$ satisfies **component efficiency** if $\sum_{i \in T} f_i(v, L) = v(T)$ for all $T \in C_L(N)$.

In van den Brink, van der Laan, and Vasil'ev (2007), four solutions are axiomatized with the use of component efficiency and one of the following four axioms, all of which concern the removal of edges.[9]

For $i = 1, \ldots, n-1$, let $(N, L(i))$ be the graph on $N$, where $L(i) = L \setminus \{\{i, i+1\}\}$ is the set of edges obtained by deleting the edge $\{i, i+1\}$ from $L$. Notice that $(v, L(i))$ is also a line-graph game for every $i \in \{1, \ldots, n-1\}$.

[8] This follows since, if $v$ is a superadditive game, then for any $S, T \subseteq N$ with $S \cap T = \varnothing$, it holds that $v^L(S \cup T) = \sum_{H \in C_L(S \cup T)} v(H) \geq \sum_{H \in C_L(S)} v(H) + \sum_{H \in C_L(T)} v(H) = v^L(S) + v^L(T)$.

[9] In van den Brink, van der Laan, and Vasil'ev (2007), these four solutions are axiomatized as so-called Harsanyi solutions of the restricted game. These latter solutions allocate the Harsanyi dividends of coalitions over the corresponding players according to a fixed weight system per coalition, which implies component efficiency.

- A point-valued solution $f$ on $\mathcal{G}^N \times \mathcal{L}^N$ is called **fair** if, for any $i = 1, \ldots, n-1$ and any $(v, L) \in \mathcal{G}^N \times \mathcal{L}^N$, it holds that $f_i(v^L) - f_i(v^{L(i)}) = f_{i+1}(v^L) - f_{i+1}(v^{L(i)})$.

- A point-valued solution $f$ on $\mathcal{G}^N \times \mathcal{L}^N$ is called **upper equivalent** if, for any $i = 1, \ldots, n-1$ and any $(v, L) \in \mathcal{G}^N \times \mathcal{L}^N$, it holds that $f_j(v^L) = f_j(v^{L(i)})$, $j = 1, \ldots, i$.

- A point-valued solution $f$ on $\mathcal{G}^N \times \mathcal{L}^N$ is called **lower equivalent** if, for any $i = 1, \ldots, n-1$ and any $(v, L) \in \mathcal{G}^N \times \mathcal{L}^N$, it holds that $f_j(v^L) = f_j(v^{L(i)})$, $j = i+1, \ldots, n$.

- A point-valued solution $f$ on $\mathcal{G}^N \times \mathcal{L}^N$ is said to have the **equal loss property** if, for any $i = 1, \ldots, n-1$ and any $(v, L) \in \mathcal{G}^N \times \mathcal{L}^N$, it holds that $\sum_{j=1}^{i}(f_j(v^L) - f_j(v^{L(i)})) = \sum_{j=i+1}^{n}(f_j(v^L) - f_j(v^{L(i)}))$.

The first property is the famous fairness property introduced by Myerson (1977) for arbitrary graph games. It states that deleting the edge between $i$ and $i+1$ hurts (or benefits) both players, $i$ and $i+1$, equally. The equal loss property can also be conceived as referring to a type of fairness, but instead of the individual payoffs of the players on the edge that is deleted, it concerns the total payoff of *all* players at both sides of the deleted edge, requiring that these total payoffs change by the same amount. Upper equivalence requires that the payoff of a player does not depend on the presence of 'downward' edges, while lower equivalence requires that the payoff of a player does not depend on the presence of 'upward' edges. Which property is most appropriate depends on the respective application, something we will discuss after the next theorem and in the following sections.

Let $f^u$, $f^\ell$, $f^e$ and $f^s$ be the point-valued solutions on $\mathcal{G}^N \times \mathcal{L}^N$ defined by $f^u(v, L) = m^u(v^L)$, $f^\ell(v, L) = m^\ell(v^L)$, $f^e(v, L) = \frac{1}{2}(m^u(v^L) + m^\ell(v^L))$, and $f^s(v, L) = Sh(v^L)$ for all $(v, L) \in \mathcal{G}^N \times \mathcal{L}^N$.[10] The solution $f^s$ is known as the Myerson value and was introduced and axiomatized by Myerson (1977) for arbitrary graph games.

**Theorem 2.2** *[van den Brink, van der Laan, and Vasil'ev (2007)] Let $f : \mathcal{G}^N \times \mathcal{L}^N \to \mathbb{R}^N$ be a component efficient solution on the class $\mathcal{G}^N \times \mathcal{L}^N$ of line-graph games. Then,*

*(i) $f$ is fair if and only if $f = f^s$.*

*(ii) $f$ is upper equivalent if and only if $f = f^u$.*

*(iii) $f$ is lower equivalent if and only if $f = f^\ell$.*

*(iv) $f$ satisfies the equal loss property if and only if $f = f^e$.*

[10] For sequencing games, Curiel et al. (1993, 1994) introduced the function $f^e$ as the $\beta$-rule, see also the next section.

Myerson (1977) already showed that, on the class of communication-graph games, the Shapley value $f^s$ is characterized by component efficiency and fairness.[11] So, the 'if' part of item (i) in Theorem 2.2 also follows immediately from Myerson (1977). The 'only if' part shows that uniqueness also holds on the (smaller) subclass of line-graph games.

As mentioned before, for any superadditive line-graph game $(v, L)$, both the lower equivalent solution $f^\ell$ and the upper equivalent solution $f^u$ are in the core of the game; hence, all convex combinations, including the equal loss solution $f^e$, are also in the core. In the rest of this paper, we focus on a superadditive class of line-graph games, called majority voting line-graph games, where the Shapley value need not belong to the core. We note, however, that when $v$ is convex, the Shapley value *does* belong to the core of the restricted game $v^L$. This follows (1) from the fact that the Shapley value of any convex cooperative game belongs to its core, and (2) from van den Nouweland and Borm (1991) (see also Algaba, Bilbao, and López, 2001), who show that when $v$ is convex, the restricted game $v^L$ is also convex.[12] What's more, when $v$ is convex, as mentioned in van den Brink, van der Laan, and Vasil'ev (2007), the restricted game $v^L$ is *almost positive*, that is, $\Delta_S(v^L) \geq 0$ whenever $|S| \geq 2$ (see also Vasil'ev 1978, 2006).[13] For such games, not only does the core contain the Shapley value, but it also coincides with the so-called *selectope* or *Harsanyi set* (Vasil'ev, 1978, 2006), denoted by $H(v)$, which is, for any game, the set of all allocations that distribute the Harsanyi dividend of any coalition $S$ over the players in $S$.[14] The following result thus follows from van den Nouweland and Borm (1991) together with, for example, Vasil'ev (2006) or Derks, Haller, and Peters (2000).[15]

**Corollary 2.3** *For any line-graph game $(v, L) \in \mathcal{G}^N \times \mathcal{L}^N$, if $v$ is convex, then the restricted game $v^L$ is almost positive and hence convex, which implies that $Sh(v^L) \in core(v^L) = H(v^L)$.*

Examples of a convex line-graph game are Ambec and Sprumont's (2002) river game, and Curiel, Pederzoli and Tijs' (1989) sequencing game. We now turn to a class of line-graph

[11] See van den Brink (2002) for a related result on the class of cooperative games. A non-cooperative implementation of the Shapley value can be found in Pérez-Castrillo and Wettstein (2001). Slikker (2007) provides a strategic implementation of the Myerson value and other graph game solutions.

[12] In van den Nouweland and Borm (1991), this is shown for all so-called *cycle-complete* graphs, being those graphs such that if there is a cycle, then the subgraph on that cycle is complete. This class obviously contains all cycle-free graphs, and thus all line graphs.

[13] In van den Brink, van der Laan, and Vasil'ev (2007), a line-graph game with an almost positive $v^L$ is called *linear-convex*.

[14] See Vasil'ev (2006) and Derks, Haller, and Peters (2000) for these results. Since the Shapley value is one way of allocating the Harsanyi dividends (namely, equally), it always belongs to the Harsanyi set.

[15] Note that, given expression (2.4), for Corollary 2.3 to hold, full convexity of $v$ is sufficient but not necessary. Instead, we may require a weaker property: $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$ for all $A, B \subseteq N$ such that $|A \setminus B| = |B \setminus A| = 1$.

games, majority voting line-graph games, that are superadditive but not convex.

## 3 Political power in majority voting line-graph games

In this section, we consider majority voting games between political parties in a parliament. We consider majority voting games where the political parties are ordered on a line according to their political preferences over, for example, economic policy, ethical issues, environmental problems, and so on.

A *majority voting situation* consists of (i) a set of political parties, (ii) a number of seats for each party, and (iii) a quota expressing how many of the seats are necessary to pass a bill. Formally, a majority voting situation is a triple $(N, s, q)$, where

1. $N = \{1, \ldots, n\}$ is the set of players representing the parties in a parliament,
2. $s = (s_i)_{i \in N}$ is the seat distribution with $s_i$ the number of seats (votes) of party (player) $i \in N$, and
3. $q$ such that $\frac{1}{2}\sum_{i \in N} s_i < q \leq \sum_{i \in N} s_i$ is the quota, being the minimum number of seats necessary to pass a ballot.

We denote the total number of seats by $w = \sum_{i \in N} s_i$. Voting power refers to the ability of the political parties to turn losing coalitions into winning ones, or vice versa. To determine these abilities, it is sufficient to know what the winning and losing coalitions are. This can be summarized by an associated *majority voting game*: a cooperative game where the worth of any coalition is either one (when it has a qualified majority) or zero (when it does not have a qualified majority). In other words, the majority voting game associated with voting situation $(N, s, q)$ is the game $(N, v)$ given by

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} s_i \geq q, \\ 0 & \text{if } \sum_{i \in S} s_i < q. \end{cases}$$

A coalition $S$ with $v(S) = 1$ is called a *winning* coalition, and a coalition $S$ with $v(S) = 0$ is called a *losing* coalition. A winning coalition $S$ is called a *minimal winning coalition* (MWC) if $v(S \setminus \{i\}) = 0$ for all $i \in S$. Two special types of players are veto players and null players. A player $i$ is a *veto player* if $v(S) = 1$ implies that $i \in S$. A player $i$ is a *null player* if $v(S \setminus \{i\}) - v(S) = 0$ for any $S$ containing $i$. Notice that, since $v(N) = 1$, every null player in a majority voting game is a non-veto player, but there can be non-veto players who are not null players.

A majority voting game is a special type of a *simple game*: a game $v$ such that $v(S) \in \{0,1\}$, $v(\varnothing) = 0$, and $v(N) = 1$.[16] It is well-known that a simple game has a non-empty core if and only if there is at least one veto player. Furthermore, the core distributes the worth $v(N) = 1$ among the veto players and assigns a zero power to all non-veto players. This follows straightforwardly from the observation that the sum of the non-negative powers of all parties in a winning coalition, including the grand coalition $N$, must be equal to one.

Two well-known power indices that measure the voting power of political parties in majority voting situations are the Shapley-Shubik index (Shapley and Shubik, 1954) and the Banzhaf index (Banzhaf, 1965). These can be obtained by applying the Shapley value (Shapley, 1953), respectively the Banzhaf value (Owen, 1975), to the associated majority voting game. Since both values assign positive power to non-null players in a majority voting game, they assign positive power to non-veto players, even when there are veto players, and thus the associated power vector is not in the core of the majority game.

We now consider a situation where the parties can be ordered linearly according to their political preferences. Without loss of generality, suppose that the parties can be indexed successively from player 1 (the most left-wing party) to player $n$ (the most right-wing party). We refer to a pair $(v, L) \in \mathcal{G}^N \times \mathcal{L}^N$ with $v$ a majority voting game, as a *majority voting line-graph game* (or, for short, *majority line-graph game*). In such a political structure, it is reasonable to suppose that only connected coalitions will form; that is, this situation can be modelled by the line-graph game $(v, \overline{L})$ where $\mathcal{I} = \{S \subseteq N \mid S = [i,j] \text{ for some } i < j\}$ is the collection of feasible coalitions. If, for some reason, two idealogical neighbours refuse to cooperate, then the cooperation restrictions can be modeled by a line-graph game $(v, L)$ with $L \subset \overline{L}, L \neq \overline{L}$.[17] As mentioned in Section 2, since $v$ is superadditive, it follows from Granot and Huberman (1982), Le Breton, Owen, and Weber (1992), Demange (1994), and Potters and Reijnierse (1995) that the core of the restricted game $v^L$ is non-empty and, specifically, that $f^u(v, L)$, $f^\ell(v, L)$, and $f^e(v, L)$ belong to the core of the restricted game $v^L$. Since the core is non-empty, $v^L$ has at least one veto player. Indeed, if $v^L(N) = 1$, since only coalitions of successive parties can form, there is at least one player who necessarily belongs to both the most left-wing MWC and the most right-wing MWC, and thus to any majority coalition.[18] Moreover, the set of veto

[16] $v(\varnothing) = 0$ requires that unanimous opposition implies rejection, $v(N) = 1$ requires that unanimous support implies acceptance. In the literature, sometimes $v(N) = 1$ is not required for a game to be a simple game.

[17] Note that even though we require that the grand coalition be winning in the majority game $v$ (that is, $v(N) = 1$), we do not require that $N$ be winning in the restricted game $v^L$. That is, $v(N) = 1$ does *not* imply $v^L(N) = 1$. For example, when $L = \varnothing$ and $s_i < q$ for all $i \in N$, $v(N) = 1$ but $v^L(N) = 0$.

[18] Notice that, since $q > w/2$, the restricted game $v^L$ is a *proper* simple game; that is, the complement of any winning coalition, including a MWC, is a losing coalition.

players is a connected coalition in the line graph $L$. If $v^L(N) = 0$, then all players are veto players and the unique core power vector assigns zero power to all players.

**Proposition 3.1** *Consider the majority line-graph game* $(v, L)$*.*

*(i) For* $h \leq k$*, let* $[1, k]$ *be the most left-wing MWC (that is,* $v([1, k]) = 1$ *and* $v([1, k-1]) = 0$*) in* $(v, \overline{L})$*, and* $[h, n]$ *be the most right-wing MWC (that is,* $v([h, n]) = 1$ *and* $v([h+1, n]) = 0$*) in* $(v, \overline{L})$*. Then* $[h, k]$ *is the set of veto players in* $v^{\overline{L}}$*.*

*(ii) For every* $L \subseteq \overline{L}$*, there is at most one component* $T = [p, q] \in C_L(N)$ *such that* $v(T) = 1$*. Let* $[p, k]$ *be the most left-wing MWC (that is,* $v([p, k]) = 1$ *and* $v([p, k-1]) = 0$*) in* $(v, L)$*, and* $[h, q]$ *be the most right-wing MWC (that is,* $v([h, q]) = 1$ *and* $v([h+1, q]) = 0$*) in* $(v, L)$*. Then* $[h, k]$ *is the set of veto players in* $v^L$*.*[19]

**Proof** Since (i) follows as a corollary from (ii) (for $T = N$, $p = 1$, and $q = n$), we only prove (ii). We show that, for all $i \in [h, k]$, if $i \notin S \subset N$, then $v^L(S) = 0$. First, note that $v(S) = 0$ if $\{h, k\} \not\subset S$ since $h$ and $k$ are veto players. It follows that, for all $i \in [h, k]$ and $S \subseteq N \setminus \{i\}$, $v^L(S) = v^L(S \cap [1, i-1]) + v^L(S \cap [i+1, n]) = 0$ since $k \notin [1, i-1]$ and $h \notin [i+1, n]$. □

**Example 3.2** Take $N = \{1, 2, 3, 4, 5, 6, 7\}$, $w = 85$ with $s_1 = s_2 = s_6 = s_7 = 10$, $s_3 = s_4 = s_5 = 15$, and $q = 60$. In the restricted game $v^{\overline{L}}$, $[1, 5]$ is the most left-wing MWC, $[3, 7]$ the most right-wing MWC, and $[3, 5] = \{3, 4, 5\}$ the set of veto players. □

In the remainder of this section, we focus on the full line graph $\overline{L}$, but the results can be straightforwardly generalized to any $L \subseteq \overline{L}$ by considering the unique winning component $T \in C_L(N)$, if any, rather than $N$. When $h = k$, it follows that $f^u(v, \overline{L}) = f^\ell(v, \overline{L}) = f^e(v, \overline{L}) = e(h)$, where $e(i) \in \mathbb{R}^N$, given by $e_i(i) = 1$ and $e_j(i) = 0$ for all $j \neq i$, is the unique core element. When $h < k$, then $f^u(v, \overline{L}) = e(k)$, $f^\ell(v, \overline{L}) = e(h)$ and $f^e(v, \overline{L}) = \frac{1}{2}(e(k) + e(h))$. So, $f^u(v, \overline{L})$ assigns full power to the most right-wing veto player, $f^\ell(v, \overline{L})$ assigns full power to the most left-wing veto player, and $f^e(v, \overline{L})$ divides the power equally between the two extreme veto players. Observe that, according to these solutions, no power is assigned to any other player, including intermediate veto players between the two extreme veto players, $h$ and $k$. This seems reasonable when the two extreme veto players are considered to be critical. When the most left-wing coalition $[1, k]$ is formed, the most right-wing veto player $k$ has the highest incentive (or lowest objection) to break away and form another MWC. So, if this player is willing to cooperate in $[1, k]$, then it can

[19]Note that a MWC in the line-graph game $(v, L)$ need *not* be a MWC in the majority game $v$.

be expected that any other player in $[1, k]$ is willing to cooperate in $[1, k]$, including any other veto player. Similarly, this holds for $h$ in the MWC $[h, n]$. The equal loss solution $f^e$, giving both players a power of $\frac{1}{2}$, seems to be an appropriate power index for a political situation before it is known whether a left-wing or right-wing majority coalition will be formed. Note that both the Shapley value and the (normalized) Banzhaf power index assign positive power to every player and, thus, they are not in the core.

The next proposition states that, in the restricted game $v^{\overline{L}}$, the dividend of each MWC in $\mathcal{I}$ is equal to 1. The dividend of any other coalition in $\mathcal{I}$ is equal to 0 or $-1$.[20]

**Proposition 3.3** *Let $v^{\overline{L}}$ be the restricted game of a majority line-graph game $(v, \overline{L})$. Then, for $S \in \mathcal{I}$, $\Delta_S(v^{\overline{L}}) = 1$ if $S$ is a MWC and $\Delta_S(v^{\overline{L}}) \in \{-1, 0\}$ otherwise.*

**Proof** First, observe that $\Delta_T(v^{\overline{L}}) = 0$ for any $T = [i, j]$ with $v[i, j] = 0$. Next, let $S = [i, j]$ be a MWC; that is, $v[i, j] = 1$ and $v(T) = 0$ for all $T \subset S$, $T \neq S$. From Theorem 2.1, it follows that $\Delta_{[i,j]}(v^{\overline{L}}) = v[i, j] - v[i, j-1] - v[i+1, j] + v[i+1, j-1] = 1-0-0+0 = 1$. Next, given a MWC $[i, j]$, consider any coalition $[i, k]$ with $k > j$. Then, $k - 1 \geq j$ and thus $v[i, k] = v[i, k-1] = 1$. Further, if $v[i+1, k-1] = 1$, then $v[i+1, k] = 1$ , and thus $\Delta_{[i,k]}(v^{\overline{L}}) = 0$ in that case. Otherwise, if $v[i+1, k-1] = 0$, since $v[i+1, k] \in \{0, 1\}$, $\Delta_{[i,k]}(v^{\overline{L}}) \in \{-1, 0\}$. Similarly, given a MWC $[i, j]$, this holds for any coalition $[h, j]$ with $h < i$. Finally, given a MWC $[i, j]$, when $S = [h, k]$ is such that $h < i < j < k$, then $v[h, k] = v[h+1, k] = v[h, k-1] = v[h+1, k-1] = 1$ and thus $\Delta_{[h,k]}(v^{\overline{L}}) = 0$. □

Since the worth $v^{\overline{L}}(N) = 1$ is equal to the sum of all dividends in the restricted game, this proposition implies that the number of coalitions with dividend equal to $-1$ is one fewer than the number of MWCs.

Notice that, in a standard majority game, it follows from formula (2.1) that $\Delta_S(v) = 1$ if $S$ is a MWC, since $v(S) = 1$ and $v(T) = 0$ for any $T \subset S$, $T \neq S$. However, as the next example shows, in a standard majority game $v$, other (winning) coalitions may also have a positive dividend and even a dividend larger than one.

**Example 3.4** Take $N = \{1, 2, 3, 4, 5\}$, $s_i = 1$ for all $i \in N$ and $q = 3$, so that $v(S) = 1$ if and only if $|S| \geq 3$. Hence, any coalition of precisely three players is a MWC and has a dividend of one. Further, any coalition of four players contains four subcoalitions of three players. By applying formula (2.2), it follows that $\Delta_S(v) = -3$ when $|S| = 4$. Finally, the grand coalition $N$ contains ten subcoalitions of three players, each with dividend 1, and five subcoalitions of four players, each with dividend $-3$. Hence, the dividends of $N$'s subcoalitions sum up to $10 + 5(-3) = -5$, which means that $\Delta_N(v) = 1 - (-5) = 6$. □

[20]This shows that the restricted game of a majority line-graph game is not almost positive.

# 4 Rewarding intermediate veto players: Hierarchical outcomes and the $\tau$-index

A power index is a mapping $f$ that assigns a power vector $f(v, L)$ to every majority line-graph game $(v, L)$. In this paper, we are mainly interested in power indices that assign a power vector in the core of the restricted game $v^L$.[21]

**Axiom 4.1** *A power index $f$ for majority line-graph games is called* **core stable** *if, for every majority line-graph game $(v, L)$, it holds that $f(v, L) \in core(v^L)$.*

From the indices considered in the previous section, $f^u$, $f^\ell$ and $f^e$ reward either one or both of the extreme veto players, but assign zero power to the intermediate veto players. As we saw, applying the Shapley value to the restricted line-graph game does reward the intermediate veto players, but it also rewards non-veto players, and thus assigns a power vector that does not belong to the core of the restricted game. (The same holds for the Banzhaf value.)

Next, we consider two types of power indices that do reward intermediate veto players without also rewarding non-veto players. Thus, unlike the Shapley and Banzhaf values, these indices are core stable.

## 4.1 Hierarchical outcomes

The upper and lower equivalent solutions for line-graph games are examples of hierarchical outcomes. Hierarchical outcomes are defined by Demange (2004) for connected cycle-free communication-graph games as specific core stable payoff or power vectors of the restricted game. In this paper, we consider line graphs (being a special type of cycle-free graphs), but do not require that the graph be connected. We directly define the hierarchical outcomes for this type of graph games, and refer to Demange (2004) for definitions on connected cycle-free graph games.

First, consider the line-graph game $(v, L)$. For each player, there is a corresponding hierarchical outcome. For $i \in N$, let $C^i_L(N) = [h_i, k_i] \in C_L(N)$ be such that $i \in C^i_L(N)$, that is, $C^i_L(N)$ is the component in $L$ that contains player $i$.

[21] We remark that a different core concept for games with restricted cooperation assigns to every graph game the set of component efficient payoff vectors such that the sum of the payoffs of all players in any connected coalition is at least equal to the worth of this coalition. For line-graph games, this gives the solution $\overline{core}(v, L) = \{x \in \mathbb{R}^N \mid \sum_{i \in C} x_i = v(C)$ for all $C \in C_L(N)$, and $\sum_{i \in S} x_i \geq v(S)$ for all $S \in \mathcal{I}(L)\}$. It is obvious that $\overline{core}(v, L) = core(v^L)$ if the game $v$ is monotonic (that is, $v(S) \leq v(T)$ if $S \subseteq T \subseteq N$) and superadditive. Since majority voting games are monotonic and superadditive, for the games considered in this paper, the two core concepts boil down to the same.

The *hierarchical outcome* corresponding to player $i$ is the payoff vector given by[22]

$$h^i_j(v,L) = \begin{cases} v([h_i,j]) - v([h_i,j-1]) & \text{if } h_i \leq j < i \\ v([j,k_i]) - v([j+1,k_i]) & \text{if } i < j \leq k_i \\ v(C^i_L(N)) - v([h_i,j-1]) - v([j+1,k_i]) & \text{if } j = i \\ 0 & \text{if } j \in N \setminus C^i_L(N). \end{cases} \tag{4.5}$$

Thus, the hierarchical outcome $h^i(v,L)$ allocates to player $j$ to the left (respectively, to the right) of $i$ the contribution of player $j$ to all players to its left (respectively, right) in its component, while player $i$ gets the surplus of its component that is left after all other players are assigned their power.

**Example 4.2** Consider the majority voting situation of Example 3.4, that is, $s_i = 1$ for all $i \in N = \{1,2,3,4,5\}$, and $q = 3$. Further, let $L = \{\{1,2\},\{3,4\},\{4,5\}\}$. Then, $h^3_1(v,L) = h^3_2(v,L) = h^3_5(v,L) = 0$, $h^3_4(v,L) = v(\{4,5\}) - v(\{5\}) = 0 - 0 = 0$, and $h^3_3(v,L) = v(\{3,4,5\}) - v(\{4,5\}) = 1 - 0 = 1$, yielding $h^3(v,L) = (0,0,1,0,0)$.

For majority line-graph games, we will refer to the power index that assigns to every majority line-graph game $(v,L)$ the hierarchical outcome $h^i(v,L)$ as the *$i$-hierarchical index*. Notice that, for $L = \overline{L}$, the $n$-hierarchical index (respectively, the 1-hierarchical index) is equivalent to the upper equivalent solution $f^u$ (respectively, the lower equivalent solution $f^\ell$) applied to this class of majority line-graph games. Further, $f^e$ is obtained as the average of these two hierarchical indices.

As Demange (2004) has showed, for superadditive games $(N,v)$, and connected cycle-free (tree) graphs $(N,L)$, each hierarchical outcome of the associated tree game $(v,L)$, $h^i(v,L)$, $i \in N$, is an extreme core allocation of the restricted game $v^L$. Since majority games are superadditive, this also holds for majority line-graph games with $L = \overline{L}$.

Herings, van der Laan, and Talman (2008) characterized the *average tree solution* which assigns to every cycle-free graph game the (component-wise) average hierarchical outcome given by

$$\overline{h}_j(v,L) = \frac{1}{|C^j_L(N)|} \sum_{i \in C^j_L(N)} h^i_j(v,L) \quad \text{for all } j \in N. \tag{4.6}$$

We refer to the power index that assigns to every majority line-graph game $(v,L)$, the average hierarchical outcome $\overline{h}(v,L)$ as the *hierarchical index*. By convexity of the core of a game and the fact that all hierarchical outcomes belong to the core of the restricted game, the hierarchical index is core stable. From now on, we denote by $Veto(v,L)$ the set of veto players in the restricted game $v^L$.

[22] With some abuse of notation, we define $[h,k]$ to be the empty set $\varnothing$ if $k < h$.

For each $i \in N$, the hierarchical outcome associated with $i$ can be expressed as follows. If $v^L(N) = 0$, then all players are veto players and every hierarchical outcome assigns zero power to all players, which is also the unique core power vector. If $v^L(N) = 1$, then the set of veto players, $Veto(v, L)$, is a subset of a component, $Veto(v, L) \subseteq S$ for some $S \in C_L(N)$. It follows then that the hierarchical outcome associated with any player outside $S$ is the zero vector. Next, consider the hierarchical outcomes associated with players in $S$. As $S$ is connected, each hierarchical outcome $h^i(v, L)$, for $i \in S$, is an extreme point in the core and hence allocates full power to a veto player. Let $Veto(v, L) = [h, k]$. For $i \in Veto(v, L)$, $h^i(v, L) = e(i)$. For $i \in S \setminus Veto(v, L)$, if $i < h$ then $h^i(v, L) = e(h)$, and if $i > k$ then $h^i(v, L) = e(k)$. In other words, the hierarchical outcome associated with a non-veto player $i \in S$ is the extreme core stable power vector that assigns full power to the veto player that is closest to $i$. This is summarized in the following theorem.

**Theorem 4.3** *Let $v^L$ be the restricted game of a majority line-graph game $(v, L)$. If $v^L(N) = 0$ then $h(v, L) = \mathbf{0}$. If $v^L(N) = 1$, then there is exactly one $S \in C_L(N)$ with $v^L(S) = 1$ and, denoting $Veto(v, L) = [h, k] \subseteq S$,*

$$h^i_j(v, L) = \begin{cases} 1 & \text{if } (j = i \in Veto(v, L)), \\ & \text{or } (j = h \text{ and } i \in S \text{ with } i < h), \\ & \text{or } (j = k \text{ and } i \in S \text{ with } i > k), \\ 0 & \text{otherwise.} \end{cases}$$

Using Theorem 4.3, it is now easy to express the hierarchical index $\bar{h}(v, L)$ given by (4.6) for majority line-graph games as follows.

**Theorem 4.4** *Let $(v, L)$ be a majority line-graph game with restricted game $v^L$ such that $v^L(N) = 1$. Further, let $[l, m]$ denote the component containing the veto players in $(v, L)$, that is, $v^L([l, m]) = 1$. We distinguish two cases.*

*(1) Suppose that $|Veto(v, L)| = 1$. Then,*

$$\bar{h}_i(v, L) = \begin{cases} 1 & \text{if } \{i\} = Veto(v, L), \\ 0 & \text{otherwise.} \end{cases}$$

*(2) Suppose that $|Veto(v, L)| > 1$. Let $Veto(v, L) = [h, k] \subseteq [l, m] \in C_L(N)$. Then,*

$$\bar{h}_i(v, L) = \begin{cases} \frac{h-l+1}{m-l+1} & \text{if } i = h, \\ \frac{m-k+1}{m-l+1} & \text{if } i = k, \\ \frac{1}{m-l+1} & \text{if } |Veto(v, L)| > 2 \text{ and } i \in [h+1, k-1], \\ 0 & \text{otherwise.} \end{cases}$$

In majority line-graph games, the hierarchical index rewards all (and only) veto players. It thus belongs to the core of the restricted game.

**Corollary 4.5** *The hierarchical index $\bar{h}$ is core stable, that is, if $v^L$ is the restricted game of a majority line-graph game $(v, L)$, then $\bar{h}(v, L) \in core(v^L)$.*

It also follows from Theorem 4.4 that the hierarchical index does not reward all veto players equally. More precisely, it rewards the two extreme veto players $h$ and $k$ more than it does the intermediate veto players in $[h + 1, k - 1]$. Further, it rewards the left (respectively, right) more than it does the right (respectively, left) extreme veto player, depending on the number of other (non-veto) players positioned to the left (respectively, to the right) of the respective veto player in the component.

Herings, van der Laan, and Talman (2008) axiomatized[23] the average tree solution by component efficiency and a component fairness property, which says that breaking an edge in a cycle-free graph game has the same per capita effect on the payoffs of the players in the two newly created components. Since, in this paper, we focus on core stable power indices, and core stability implies component efficiency, to characterize the hierarchical index we can do with a weaker component fairness axiom, where we only consider the breaking of edges between two veto players. Recall that $L(i) = L \setminus \{\{i, i+1\}\}$ is the set of edges obtained by deleting the edge $\{i, i+1\}$ from $L$.

**Axiom 4.6** *A power index $f$ for majority line-graph games is called* **component veto fair** *if, for every majority line-graph $(v, L)$ with $Veto(v, L) = [h, k]$, and for all $i \in [h, k-1]$, it holds that*

$$\frac{1}{|C^i_{L(i)}(N)|} \sum_{j \in C^i_{L(i)}(N)} (f_j(v, L) - f_j(v, L(i))) = \frac{1}{|C^{i+1}_{L(i)}(N)|} \sum_{j \in C^{i+1}_{L(i)}(N)} (f_j(v, L) - f_j(v, L(i)))$$

The hierarchical index $\bar{h}$ is the only power index that satisfies core stability and component veto fairness.

**Theorem 4.7** *Let $f$ be a power index for majority line-graph games. Then, $f$ is core stable and component veto fair if and only if $f = \bar{h}$.*

**Proof** The 'if' part follows from the more general result in Herings, van der Laan, and Talman (2008) and Corollary 4.5. Uniqueness could be showed in a way similar to that in Herings, van der Laan, and Talman (2008). But since it is not a corollary of their result,

[23] A strategic implementation of the hierarchical outcomes and their average can be found in van den Brink, van der Laan, and Moes (2013).

and we use the stronger core stability but weaker component veto fairness property, we give the uniqueness proof for completeness.[24]

Let $f$ be a power index for majority line-graph games that satisfies core stability and component veto fairness. If $v^L(N) = 0$, then core stability implies that $f(v, L) = \mathbf{0}$.

Now, suppose that $v^L(N) = 1$. By core stability, we have

$$f_i(v, L) = 0 \quad \text{for all } i \in N \setminus Veto(v, L). \tag{4.7}$$

Let $Veto(v, L) = [h, k] \subseteq [l, m] \in C_L(N)$. We distinguish the following two cases.

When $|Veto(v, L)| = 1$, then $f_j(v, L) = 1$ for $\{j\} = Veto(v, L)$ follows from core stability (particularly, from efficiency). This, together with the $n - 1$ equations in (4.7), determines $f(v, L)$.

Suppose that $|Veto(v, L)| > 1$. We proceed by induction on the number of edges $|L|$. When $L = \varnothing$, core stability implies that the core consists of a unique vector, the zero vector. Hence, $f_i(v, L) = 0$ for all $i \in N$. Assume that $f(v, L')$ is determined for all $L'$ with $|L'| < |L|$.

By component veto fairness, we have, for all $i \in [h, k - 1]$,

$$\frac{1}{|C^i_{L(i)}(N)|} \sum_{j \in C^i_{L(i)}(N)} (f_j(v, L) - f_j(v, L(i))) = \frac{1}{|C^{i+1}_{L(i)}(N)|} \sum_{j \in C^{i+1}_{L(i)}(N)} (f_j(v, L) - f_j(v, L(i))) . \tag{4.8}$$

Next, by core stability and (4.7), we have

$$\sum_{i \in [h,k]} f_i(v, L) = 1. \tag{4.9}$$

Since $f(v, L(i))$ in (4.8) is determined by the induction hypothesis, there are $n-(k-h+1) = n-k+h-1$ equations of type (4.7), and $k-h$ equations of type (4.8). Hence, together with the equation in (4.9), there are $n$ independent equations that determine the $n$ unknowns $f_i(v, L)$, $i \in N$. Since the hierarchical index satisfies the two axioms, $f$ must be equal to $\bar{h}$. □

## 4.2 The $\tau$-index

An interesting power index for political line-graph games is the solution that assigns *equal* power to all veto players and zero power to all non-veto players. It turns out that this

[24]We remark that the proof follows the one in Herings, van der Laan, and Talman (2008), except that we have fewer equations from component veto fairness, but more equations from core stability.

power index is obtained by applying the $\tau$-value (Tijs, 1981) to the associated restricted game.

First, observe that the restricted game of a majority line-graph game is quasi-balanced and applying the $\tau$-value gives the following power vectors.

**Theorem 4.8** *Let $v^L$ be the restricted game of a majority line-graph game $(v, L)$. Then, $v^L \in QB^N$. If $v^L(N) = 0$, then $\tau(v^L) = \mathbf{0}$. If $v^L(N) = 1$, then*

$$\tau_i(v^L) = \begin{cases} \frac{1}{|Veto(v,L)|} & \text{if } i \in Veto(v, L), \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** Let $v^L$ be the restricted game of a majority line-graph game $(v, L)$. If $v^L(N) = 0$, then $M(v^L) = m(v^L) = \mathbf{0}$, and the result follows. Now, suppose that $v^L(N) = 1$. Then,

$$M_i(v^L) = \begin{cases} 1 & \text{if } i \in Veto(v, L), \\ 0 & \text{otherwise.} \end{cases}$$

Next, consider a player's minimal right and distinguish the following two cases:

(1) Suppose that $|Veto(v, L)| > 1$. We then have

$$m_i(v^L) = 0 \text{ for all } i \in N.$$

(2) Suppose that $|Veto(v, L)| = 1$. We then have

$$m_i(v^L) = \begin{cases} 1 & \text{if } i \in Veto(v, L), \\ 0 & \text{otherwise.} \end{cases}$$

In both cases, $v^L$ is quasi-balanced and $\tau(v^L)$ is as stated in the theorem. □

Since the $\tau$-value allocates the whole power in the game over the veto players, and none to non-veto players, we immediately have the following corollary.

**Corollary 4.9** *Let $v^L$ be the restricted game of a majority line-graph game $(v, L)$. Then, $\tau(v^L) \in core(v^L)$.*

Next, we consider the power index that assigns to every majority line-graph game $(v, L)$, the $\tau$-value of the restricted game $v^L$: $\tau(v, L) = \tau(v^L)$. We refer to this power index as the *$\tau$-index*. We axiomatize this power index with the following two axioms. The first is core stability as defined in the previous subsection, which requires that the index always assign a power vector that belongs to the core of the restricted game. By Corollary 4.9, this is obviously satisfied for the $\tau$-index.

It is straightforward to show that Theorem 2.2.(i) also holds for the specific class of majority line-graph games; that is, the Shapley value is the unique solution for majority line-graph games that satisfies component efficiency and fairness. Since every core stable solution satisfies component efficiency and the Shapley value is not core stable, this implies that there is no core stable solution that satisfies fairness.[25] It turns out that the $\tau$-index satisfies a weaker fairness property where equal gains or losses are required only when one breaks an edge between veto players.

**Axiom 4.10** *A power index $f$ for majority line-graph games is called* **veto fair** *if, for every majority line-graph $(v, L)$ with $Veto(v, L) = [h, k]$, it holds that*

$$f_i(v, L) - f_i(v, L(i)) = f_{i+1}(v, L) - f_{i+1}(v, L(i)) \quad \textit{for all } i \in [h, k-1].$$

The $\tau$-index is the only power index that satisfies core stability and veto fairness.

**Theorem 4.11** *Let $f$ be a power index for majority line-graph games. Then $f$ is core stable and veto fair if and only if $f = \tau$.*

**Proof** From Corollary 4.9 it follows that the $\tau$-index is core stable. Veto fairness follows since (i) if $v^L(N) = 0$ then $v^{L'}(N) = 0$ for all $L' \subset L$ and $\tau(v, L) = \mathbf{0}$, and (ii) if $v^L(N) = 1$ then, for every $i, i+1 \in Veto(v, L)$,

$$\tau_i(v, L) - \tau_i(v, L(i)) = \frac{1}{|Veto(v, L)|} - 0 = \tau_{i+1}(v, L) - \tau_{i+1}(v, L(i)).$$

The proof of uniqueness follows a well-known structure. To prove uniqueness, suppose that $f$ is a power index for majority line-graph games that satisfies core stability and veto fairness. Core stability implies that $f(v, L) = \mathbf{0}$ if $v^L(N) = 0$.

Next, suppose that $v^L(N) = 1$. Core stability implies that $f_i(v, L) = 0$ for all $i \in N \setminus Veto(v, L)$.

Recall that the set of veto players is a set of connected players $[h, k]$, see Proposition 3.1.

If $|Veto(v, L)| = 1$, then by component efficiency $f_i(v, L) = 1$, with $\{i\} = Veto(v, L)$, is determined by core stability (which implies component efficiency).

If $|Veto(v, L)| > 1$, then we prove uniqueness by induction on $|L|$. If $|L| = 0$, that is, $L = \varnothing$, then, since there are at least two veto players, $core(v^L)$ is a singleton with the zero vector as its only element, and thus $f_i(v, L) = 0$ for all $i \in N$.[26] Proceeding

[25] In van den Brink, Núñez, and Robles (2021), a similar observation is made about assignment games. They characterize the $\tau$-value in such games, where it coincides with the fair division point (Thompson, 1981), see Núñez and Rafels (2002).

[26] Recall also the observation in footnote 16.

by induction, assume that $f(v, L')$ is uniquely determined for all $L'$ with $|L'| < |L|$. Let $Veto(v, L) = [h, k]$.

Veto fairness implies that

$$f_i(v, L) - f_i(v, L(i)) = f_{i+1}(v, L) - f_{i+1}(v, L(i)) \text{ for all } i \in [h, k-1]. \tag{4.10}$$

Core stability implies that

$$f_i(v, L) = 0 \text{ for all } i \in N \setminus [h, k], \tag{4.11}$$

and thus

$$\sum_{i \in [h,k]} f_i(v, L) = 1. \tag{4.12}$$

Since $f(v, L(i))$ is determined by the induction hypothesis, (4.10), (4.11) and (4.12) give $(k - 1 - h + 1) + (n - k + h - 1) + 1 = n$ independent equations in the $n$ unknown powers $f_i(v, L)$, $i \in N$, implying that $f(v, L)$ is uniquely determined. Since the $\tau$-index satisfies the axioms, $f$ must be equal to $\tau$. □

Notice that in the axiomatization of the $\tau$-index in Theorem 4.11, we used a similar weakening of fairness as we did with component fairness for the hierarchical index $\bar{h}$ in Theorem 4.7. Instead of deleting any edge between two neighbours, we only consider deleting edges between veto players. But also notice the different impact of these results. The standard component fairness axiom is compatible with core stability, but requiring stronger core stability, instead of component efficiency, allows us to use the weaker component veto fairness. The standard fairness axiom is incompatible with core stability, but the weaker veto fairness is compatible with core stability and the two axioms together characterize the $\tau$-index.

## 4.3 Illustration

We conclude this section by visually comparing the power indices discussed in this paper using a majority line-graph game $(v, L)$ with three veto players, $|Veto(v, L)| = 3$, and any number of other non-veto players. Suppose that the veto set is part of a connected component with $v^L(N) = 1$, that is, $Veto(v, L) \subseteq [l, m] \in C_L(N)$. As the power indices we are concerned with are core stable, all of them assign zero power to players outside $Veto(v, L)$ and, hence, we can focus on the three veto players.

The simplex in Figure 2 illustrates the core of the game and thus any point in $ABC$ is a core allocation (and any point outside it is not). Points $A$ and $C$ are where the left and right, respectively, extreme veto players are assigned full power. Similarly, at $B$, the

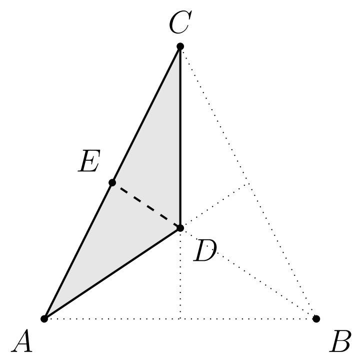


Figure 1: Core of a majority line-graph game $(v, L)$ with three veto players, where $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, $D = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and $E = (\frac{1}{2}, 0, \frac{1}{2})$.

intermediate veto player gets full power. In point $D$, all veto players get equal power. Our interest, in this paper, has been in the power allocations in the shaded core area, $ADC$. As we know from Theorem 4.8, the $\tau$-index allocation depends only on the number of veto players, since it allocates the full power of one equally among all veto players. Thus, regardless of the size of $N \setminus Veto(v, L)$ or the edges outside $[l, m]$, $\tau(v, L) = D$, where each veto player is assigned equal power. Similarly, regardless of $N \setminus Veto(v, L)$ or the edges outside $[l, m]$, $f^u(v, L) = C$, $f^l(v, L) = A$, and $f^e(v, L) = E$, since $f^u$ (respectively, $f^l$) assigns full power to the right (respectively, left) extreme veto player, while $f^e$ equally splits the power between them.

Now, suppose that $Veto(v, L) = [j, j+1, j+2]$ and consider the hierarchical indices. Any $i$-hierarchical index where $l \leq i \leq j$ allocates full power to the left-most extreme veto player $j$ and hence in that case $h^i(v, L) = e(j) = A$. Similarly, for $j + 2 \leq i \leq m$, the $i$-hierarchical index allocates full power to the right-most veto player $j + 2$, and thus in that case $h^i(v, L) = e(j + 2) = C$. When $i = j + 1$, $h^i(v, L) = e(j + 1) = B$.

Next, consider the hierarchical index $\bar{h}$. This index may assign any allocation in triangle $ADC$, *excluding* the sides $AC$, $AD$, and $DC$, but including point $D$. The location of $\bar{h}$ depends on the size, and distribution along the line, of $[l, m] \setminus Veto(v, L)$. If the veto component coincides with the veto set, $[l, m] = Veto(v, L)$, then $\bar{h}(v, L) = D$. Now, as one starts adding an increasing *equal* number of non-veto players to the left of $j$ and to the right of $j + 2$, the hierarchical index moves north-west along $DE$, tending towards $E$ as more players are added on both sides (but never reaching $E$). When the number of players to the left of the left-most extreme veto player $j$ is higher than the number of players to the right of the right-most extreme veto player $j + 2$, then $\bar{h}$ lies in the interior of $ADE$. When the reverse is the case, $\bar{h}$ lies in the interior of $CDE$. This means that the $\bar{h}$ index of, particularly, the extreme veto players depends crucially on the respective left/right-side non-veto players. Under $\bar{h}$, we might think of the set of non-veto players on the respective left/right side as giving leverage for the respective left/right-most extreme veto player.

# 5 Concluding remarks

Our main focus in this paper has been on majority games where players can be ordered linearly according to their political preferences. The core of such majority line-graph games is non-empty, and it consists of allocations that reward only the veto players in the game. Two of these allocations, which reward only the two extreme veto players, are those recommended by the upper and lower equivalent solutions. Instead, here, we focused on two core-stable point-valued solutions, the hierarchical index $\bar{h}$ and the $\tau$-index, that reward positively *all* veto players, both extreme and intermediate.[27] More precisely, according to the $\tau$-index, all veto players are equally powerful, while, according to the hierarchical index $\bar{h}$, the extreme (left and right-wing) veto players are no less (and are often more) powerful than the intermediate veto players, depending on the number of players on each side of these extreme veto players. As we saw in Section 4, for majority line-graph games, the two indices can be characterized by core stability with an additional weak (veto) fairness axiom. For the hierarchical index $\bar{h}$, this is the component veto fairness axiom—a weaker version of Herings et al.'s (2008) component fairness property—while, for the $\tau$-index, this is the veto fairness axiom—a weaker version of Myerson's (1977) fairness property. Interestingly, while—in contrast to the component fairness property—the standard fairness axiom is not compatible with core stability, its weaker veto version is; indeed, these two axioms characterize the $\tau$-index on majority line-graph games.

Some years ago in a private meeting, Stefan Napel remarked that solutions that assign non-zero power to null players are particularly interesting. We note here that majority line-graph games make the hierarchical index $\bar{h}$ and the $\tau$-index interesting in this sense. Notice that null players in the majority game $v$ need *not* be null players in the restricted game $v^L$. In fact, players who are null players in $v$ might be *veto* players in $v^L$.[28] Such players may thus be assigned positive power by a core-stable power index, and are always assigned positive power by the core-stable hierarchical index $\bar{h}$ and $\tau$-index. This elevated status of null players in $v$ is a consequence of their position in the (linear) ordering on the players and the cooperation possibilities between them captured by the line graph $(N, L)$. Indeed, such a linear ordering, reflecting alliance possibilities among the member states, might explain the Luxembourg 'gaffe' in the 1958–1972 iteration of the EU Council

[27]Another solution, specifically defined for line-graph games, is the spectrum value introduced in Álvarez-Mozos, Hellman, and Winter (2013). However, as the Shapley value and the Banzhaf value, this solution is also not core stable for majority voting line-graph games.

[28]Note, however, that null players in $v$ can only be *intermediate*, and never extreme, veto players in $v^L$.

of Ministers.[29]

Majority line-graph games are also interesting because, in such games, a number of popular core-stable solutions agree in their recommendations. More precisely, as Potters and Reijnierse (1995) have showed, in superadditive *tree* games, (1) the *bargaining set* (Aumann and Maschler, 1964) coincides with the core, and (2) the *kernel* (Davis and Maschler, 1965), which is a subset of the bargaining set, is a singleton that contains only the *nucleolus* (Schmeidler, 1969). Since the restricted game $v^L$ is superadditive and line graphs are trees, these results also hold for majority line-graph games. In fact, it is easy to see that, in these games, the nucleolus coincides with the $\tau$-index.[30]

We close by referring back to an observation about the hierarchical index $\bar{h}$ noted at the end of the preceding section. Recall that, according to the hierarchical index, the power of the left and right-wing extreme veto players depends on the number of non-veto players occupying positions on either side. Put succinctly, the higher the number of non-veto players in the respective 'flank' of an extreme veto player—that is, the non-veto players to the left (right) of the left (right) extreme veto player—the more powerful that veto player is. Thus, while non-veto players have no power according to the hierarchical index $\bar{h}$, they do affect the power of their closest extreme veto player. We might then say that, while powerless according to $\bar{h}$, the non-veto players in the winning component of a line-graph game have *leverage* over the $\bar{h}$-power of the two extreme veto players. The same does not hold for the $\tau$-index, which rewards all veto players equally and according to which the power of a veto player is not sensitive to the number of non-veto players in the winning component. These observations suggest an interesting line for future research: the formulation of leverage measures that can capture the effect players have over the power of other players, given an underlying power index.[31] In the case of core-stable power indices, power*less* (that is, non-veto) players may nevertheless hold leverage over power*ful* (that is, veto) players. In such cases then, we would need to keep the ideas of 'power' and 'leverage' conceptually and formally distinct. Developing these ideas in more detail, however, is left for another occasion.

[29]Famously, Luxembourg was a null player in the Council at that time, which consisted of Germany, Italy, France, The Netherlands, Belgium, and Luxembourg with weights of 4, 4, 4, 2, 2, and 1, respectively, and a quota of 12 (Felsenthal and Machover, 1997). It is easy to verify that in all orders where Luxembourg 'separates' the big four-weight countries and the small two-weight countries (for example, take the order Belgium, Italy, Luxembourg, France, The Netherlands, and Germany), Luxembourg is an intermediate veto player in the respective majority line-graph game.

[30]This also follows from the fact that the kernel of cooperative games satisfies symmetry (it rewards symmetric players equally; see Maschler, 1992, 621) and that veto players in line-graph games are symmetric.

[31]See Casajus (2021) for such a recent contribution with respect to the Shapley value for cooperative games, where what we call a player's 'leverage' is called a player's 'second-order productivity'.

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